Taking Kaluza seriously leads to a non-gauge-invariant electromagnetic theory in a curved space-time

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abstract

Kaluza's metric with the cylinder condition is considered without the weak gravitational field approximation. It is shown that these hypoteses lead to a non-gauge-invariant electromagnetic theory in a curved space-time. The problem of electro-gravitational unification is considered from this point of view.

1 Introduction.

In the early Twenty's, Theodor Kaluza showed that by adding a fifth dimension to space-time electromagnetism could be geometrized side by side with gravitation, namely that provided one assumed proportionality of the electromagnetic potentials to the mixed components ($\mu 5$) of the metric tensor, one could derive the set of the equations of the Einstein-Maxwell theory(1). Inasmuch as it gives nothing more that electromagnetism *and* gravitation, with no effective extension of the physical picture, Kaluza's theory is subject to the obvious remark that it does not produce anythig new, thus representig a purely formal transcription of already existing theories.

Kaluza's results were however obtained in the weak field approximation. A few years later Oskar Klein re-discovered the fifth dimention, and, with an appropriate choice of the metric, was able to re-obtain Kaluza's results(2). In this paper we take the attitude to stick to Kaluza's original choice for the metric, but to avoid taking the weak field approximation from the start. Calculations shall be carried out exactly throughout, and a weak field approximation shall be considered, as a final step, only for the electromagnetic field.

Thus rephrased, Kaluza's framework proves richer than commonly believed. It turns out that it does not merely reproduce the Einstein-Maxwell theory, but describes a theory of the electromagnetic field in an actually curved space-time, with an extra electro-gravitational coupling. A notable feature is a formal breaking of the electromagnetic gauge invariance, not unlike the one taking place, for instance, in the transition to superconductivity, exhibited by the form of the coupling and the appearance of a mass term in the wave equation. I thought it worthwhile to present this result not as much for its direct physical interest but for its formal resemblance to other cases of symmetry breaking.

2 Hypotheses and assumptions.

Following Kaluza's basic idea we deal with a 5-dimentional world to which we attribute a 5-dimensional Riemann geometry. We also assume that the components of every tensor are independent from x^5, that is $\partial_5 = 0$.

From now on, all our tensors will be projected on the 4-dimensional space-time, and we will use the convention that every tensor with a subscript 5 on its left lives on the 5-dimensional space, while the same tensor with subscript 4 lives on the 4-dimensional space. Latin capital letter indexes run from 1 to 5, greek indexes from 1 to 4.

With these hypotheses we write the 5-dimensional metric as

$$g_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 2aA_\alpha \\ \hline 2aA_\beta & 1 \end{array}\right) \tag{1}$$

with a a dimensionless constant.

We define the following two quantities

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{2}$$

$$\Sigma_{\alpha\beta} = \partial_\alpha A_\beta + \partial_\beta A_\alpha \tag{3}$$

where, according to Kaluza's interpretation, $F_{\alpha\beta}$ is the electromagnetic field.

For the mathematical form of the 5-source tensor we generalize the 4-Energy-Momentum tensor of the matter, so we can write

$$T_{AB} = \sigma_0 u_A u_B \tag{4}$$

where σ_0 is the proper inertial mass density.

We may now define the following two quantities :
1) $\mu_0 \equiv \sigma_0$, where μ_0 is the proper gravitational charge density;
2) $\frac{ac}{2}\rho_0 \equiv \sigma_0 u_5$, where ρ_0 is the proper electric charge density.

We observe that

i) since σ_0 is not negative and u_5 may assume every value we obtain that the gravitational charge cannot be negative, while the electric one can be positive, negative, or null;

ii) writing $\rho_0 = \frac{2}{ac}\mu_0 u_5$, we can see massless charged particles cannot exist.

Projected on the 4-dimensional space-time and using these definitions, the 5-source tensor is

$$_5T_{\alpha\beta} = \mu_0 \, {}_5u_\alpha \, {}_5u_\beta \tag{5}$$

$$_5T_{5\beta} = \frac{ac}{2} \, {}_5u_\beta \, \rho_0 \tag{6}$$

Setting $G = 1$, we finally write the Einstein 5-dimensional equations

$$G_{AB} + \frac{8\pi}{c^2} T_{AB} = 0 \tag{7}$$

where A and B cannot be both equal to 5. These, projected on the 4-dimensional space-time, split in a set of tensor equations of order 2, and a set of vector equations

$$
\begin{aligned}
{}_5G_{\alpha\beta} + \frac{8\pi}{c^2}\mu_0 \, {}_5u_\alpha \, {}_5u_\beta &= 0 \qquad (8)\\
{}_5G_{5\beta} + a\frac{4\pi}{c} \, {}_5u_\beta \rho_0 &= 0 \qquad (9)
\end{aligned}
$$

3 Calculus and approximations

Now we start the calculus of the Levi-Civita connection and of the Riemann, Ricci and scalar curvatures; from the last two we obtain the Einstein tensor.

For the Levi-Civita connection we find

$$
\begin{aligned}
{}_5\Gamma_{\mu,\alpha\beta} &= {}_4\Gamma_{\mu,\alpha\beta} \qquad (10)\\
{}_5\Gamma_{\mu,\alpha 5} &= aF_{\alpha\mu} \qquad (11)\\
{}_5\Gamma_{5,\alpha\beta} &= a\Sigma_{\alpha\beta} \qquad (12)
\end{aligned}
$$

For the Einstein tensor we obtain

$$
\begin{aligned}
{}_5G_{\alpha\beta} &= {}_4G_{\alpha\beta} + O_{\alpha\beta}(a^2) + O'_{\alpha\beta}(a^4) \qquad (13)\\
{}_5G_{5\beta} &= a({}_4\nabla_\mu F_\beta{}^\mu - A_\beta \, {}_4R) + \Omega_\beta(a^3) + \Omega'_\beta(a^5) \qquad (14)
\end{aligned}
$$

where $O_{\alpha\beta}(a^2)$ and $O'_{\alpha\beta}(a^4)$ are two tensors that behave as a^2 and a^4, when $a \longrightarrow 0$; $\Omega_\beta(a^3)$ and $\Omega'_\beta(a^5)$ are two vectors that behave as a^3 and a^5 in the same limit.
We then make the approximation according to which the dimensionless constant a is very small, with the physical interpretation that the electromagnetic field is not strong enough as to contribute to the space-time curvature.

In this limit, we obtain

$$
{}_5u_\mu =_4 u_\mu \qquad (15)
$$

so that

$$
\begin{aligned}
{}_5T_{\alpha\beta} &= {}_4T_{\alpha\beta} \qquad (16)\\
{}_5T_{5\beta} &= \frac{ac}{2} \, {}_5u_\beta\rho_0 = \frac{ac}{2} \, {}_4u_\beta\rho_0 = \frac{ac}{2} j_\beta \qquad (17)\\
& \qquad (18)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_5G_{\alpha\beta} &= {}_4G_{\alpha\beta} \qquad (19)\\
{}_5G_{5\beta} &= a({}_4\nabla_\mu F_\beta{}^\mu - A_\beta \, {}_4R) \qquad (20)
\end{aligned}
$$

In this approximation, the Einstein's equations become

$$
\begin{aligned}
{}_4G_{\alpha\beta} + \frac{8\pi}{c^2} \, {}_4T_{\alpha\beta} &= 0 \qquad (21)\\
a({}_4\nabla_\mu F_\beta{}^\mu - A_\beta \, {}_4R + \frac{4\pi}{c} j_\beta) &= 0 \qquad (22)
\end{aligned}
$$

Omitting the subscript 4, we finally have Einstein's equations for the gravitational field

$$G^{\alpha\beta} = -\frac{8\pi}{c^2} T^{\alpha\beta} \tag{23}$$

separated from the electromagnetic one as our approximation requires, and a new kind of 4-vector equations

$$\nabla_\mu F^{\mu\beta} + A^\beta R = \frac{4\pi}{c} j^\beta \tag{24}$$

We observe that :

i) In the electromagnetic vacuum one has

$$\nabla_\mu F^{\mu\beta} + A^\beta R = 0 \tag{25}$$

linear in the electromagnetic potential.

ii) taking the covariant quadridivergence of this equation gives a subsidiary condition of the form

$$\nabla_\mu (A^\mu R) = 0 \tag{26}$$

Indeed, from the identity

$$\nabla_\mu \nabla_\lambda S_{\alpha\beta} - \nabla_\lambda \nabla_\mu S_{\alpha\beta} \equiv S_{\alpha\nu} R^\nu{}_{\beta\lambda\mu} + S_{\nu\beta} R^\nu{}_{\alpha\lambda\mu} \tag{27}$$

where $R^\nu{}_{\beta\lambda\mu}$ is the Riemann curvature and $S_{\alpha\beta}$ a generic tensor, one obtains with a little algebra

$$\nabla_\mu \nabla_\alpha S^{\mu\alpha} \equiv \nabla_\alpha \nabla_\mu S^{\mu\alpha} \tag{28}$$

If $S_{\alpha\beta}$ is an antisimmetric tensor one then has

$$\nabla_\mu \nabla_\alpha S^{\mu\alpha} \equiv 0 \tag{29}$$

This identity and the continuity equation for the electric current $\nabla_\alpha j^\alpha = 0$ lead to the subsidiary condition.

With this subsidiary condition, the field equations read

$$\nabla_\mu F^{\mu\beta} + A^\beta R = 0 \tag{30}$$
$$\nabla_\mu (A^\mu R) = 0 \tag{31}$$

and describe a non-gauge-invariant 4-vector massive field.

In the gravitational vacuum we have $R \equiv 0$ and the equations for the electromagnetic field reduce to

$$\nabla_\mu F^{\mu\alpha} = \frac{4\pi}{c} j^\alpha \tag{32}$$

Furthermore, we have the identities

$$\epsilon^{\alpha\beta\gamma\delta} \nabla_\gamma F_{\alpha\beta} = 0 \tag{33}$$

i.e. the whole set of Maxwell's inhomogeneous and homogeneous equations in a curved space-time.

4 Summary.

As we have already observed, we have a set of 5 equations for the electromagnetic field in the matter

$$\begin{aligned} \nabla_\mu F^{\mu\alpha} + A^\alpha R &= \frac{4\pi}{c} j^\alpha && (34) \\ \nabla_\mu (A^\mu R) &= 0 && (35) \end{aligned}$$

This a Proca-like system of equations describing a non-gauge- invariant 4-vector massive field. This can be interpretated in this way: when an electromagnetic field propagates in a space-time with a non-vanishing scalar curvature, it couples with the gravitational one and this coupling brakes the gauge simmetry, giving a mass to the photon.

However, since R, calculated in a fixed point, is different from 0 if and only if the proper inertial mass density does not vanish in that point, and since when $R \neq 0$ the matter undergoes a gravitational collapse, one has a hint as to why, even assuming the validity of the present framework, the non-gauge-invariant term does not show up in Maxwell's equations and why the photon mass has always been found equal to 0.

The non-gauge-invariance of this theory is also interesting for a discussion about the problem of unification. Following Pauli, every theory which is generally covariant and gauge invariant can be formulated in Kaluza's form; although this theory is generally covariant, it is not gauge invariant, and thus, it cannot be formulated in Kaluza's form unless this possibility is warranted by its own structure.

For a more general discussion about unified theories we can recall Lichnerowicz's definition of a unified theory[(3)]:

"A theory is unified in a broad sense if, in the representation of the fields and in the form of the equations, it attributes symmetrical roles to the two fields; in particular, in the conceptions of general relativity, the two fields emanate from the same geometry. A theory is unified in a strict sense if the exact equations govern a non-decomposable hyperfield, and they can only approximately be decomposed into two field equations when one of the fields dominates the other."

From this point of view, both Kaluza's theory and this extension satisfy Lichnerowicz's definition of a unified theory in the broad sense, since the gravitational and the electromagnetic field emanate from the same geomerty. However, Kaluza's original theory is nothing more than the Einstein-Maxwell theory, while, in this extension, we have Einstein and Maxwell theories in a curved space-time as the approximated form of a 4-tensor theory and of a 4-vector one which are two different space-time projections of a same 5-tensor theory.

From these considerations we obtain, as Pauli already said[(4)], that the Kaluza's theory is not a unified theory in the strict sense; but from the same considerations we also obtain that this theory is a unified theory in the strict sense.

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